Math 112: Introductory Real Analysis

Last time: convergence of sequences

Today: subsequences and Cauchy sequences

Def Given a sequence $\{p_n\}$, a *subsequence* of $\{p_n\}$ is a sequence of the form

$$\{p_{n_k}\}_{k=1}^{\infty} \quad \text{where } n_1 < n_2 < n_3 < \cdots \text{ is a sequence of positive integers.}$$

Note, $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.

Thm If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point in X.

proof) Let $E \subseteq X$ be the range of $\{p_n\}$.

If E is finite, then there is a point $p \in E$ that appears infinitely many times in the sequence $\{p_n\}$, so we can take the subsequence to be the constant subsequence

$\{p_{n_k}\}$ ~~where~~ with $n_1 < n_2 < n_3 < \cdots$ such that $p_{n_1} = p_{n_2} = \cdots = p$.

If E is infinite, E has a limit point $p \in X$.

Choose n_1 so that $d(p, p_{n_1}) < 1$.

Having chosen $n_1, \ldots, n_{i-1}$, choose n_i so that $n_i > n_{i-1}$ and $d(p, p_i) < \frac{1}{i}$.

Then $\{p_{n_k}\}$ converges to p ∎

Cor Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

If $\{p_n\}$ has a subsequence converging to p, we say p is a <u>subsequential limit</u> of $\{p_n\}$. Note, subsequential limits are not unique!

<u>Thm</u> The set of all subsequential limits of a sequence $\{p_n\}$ in a metric space X forms a closed subset of X.

proof) Let E^* be the set of all subsequential limits of $\{p_n\}$ and let $q \in (E^*)'$. We have to show $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$ (If no such n_1 exists, there is nothing to prove).

Set $\delta := d(q, p_{n_1})$.

Suppose $n_1, \cdots, n_{i-1}$ are chosen.

Since $q \in (E^*)'$, there is $x \in E^*$ such that $d(x, q) < \frac{\delta}{2i}$,

and since $x \in E^*$, there is $n_i > n_{i-1}$ such that $d(p_{n_i}, x) < \frac{\delta}{2i}$.

Thus $d(p_{n_i}, q) \leq \frac{\delta}{i}$ for $i = 1, 2, 3, \cdots$,

i.e. $\{p_{n_i}\}$ converges to q. Hence $q \in E^*$. ■

Def A sequence $\{p_n\}$ in a metric space is said to be a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N such that

$$d(p_n, p_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

Note, the limit is not explicitly involved in the definition.

Thm
(a) In any metric space X, every convergent sequence is a Cauchy sequence.
(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in it, then $\{p_n\}$ converges to some point in X.
(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

So, in compact metric spaces and in $\mathbb{R}^k$, convergent sequences are equivalent to Cauchy sequences.

Def A metric space in which every Cauchy sequence converges is said to be complete.

The above theorem says that compact metric spaces and Euclidean spaces are complete. We also know that every closed subset of a complete metric space is complete.

4/ proof)

(a) If $\lim_{n\to\infty} p_n = p$ and if $\varepsilon > 0$, there is an integer N such that

$d(p, p_n) < \frac{\varepsilon}{2}$ for all $n \geq N$.

Hence $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

for all $n, m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

(b) Let $\{p_n\}$ be a Cauchy sequence in the compact space X.

For $n = 1, 2, 3, \cdots$, let $E_n := \{p_n, p_{n+1}, p_{n+2}, \cdots\}$.

Then $\overline{E}_1 \supset \overline{E}_2 \supset \cdots$ and each of $\overline{E}_n$ is compact (being a closed subset of a compact space). It follows that $\bigcap_{n=1}^{\infty} \overline{E}_n$ is non-empty.

We claim that $\bigcap_{n=1}^{\infty} \overline{E}_n$ consists of a single point.

(This is because, if there were $p, q \in \bigcap_{n=1}^{\infty} \overline{E}_n$ with $p \neq q$, say $\delta := d(p,q) > 0$, there is an integer N such that $d(p_n, p_m) < \frac{\delta}{3}$ for all $n, m \geq N$, and since $p, q \in \overline{E}_N$, there are some $p_n, p_m \in E_N$ with $d(p, p_n) < \frac{\delta}{3}$ and $d(q, p_m) < \frac{\delta}{3}$,

but then we have $\delta = d(p,q) \leq d(p, p_n) + d(p_n, p_m) + d(p_m, q) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$,

a contradiction.)

Let $p \in \bigcap_{n=1}^{\infty} \overline{E}_n$ be the unique point. Then, for any $\varepsilon > 0$, there is N such that $d(p_n, p_m) < \frac{\varepsilon}{2}$ for any $p_n, p_m \in E_N$. Moreover, since $p \in \overline{E}_N$, there is $p_n \in E_N$ such that $d(p, p_n) < \frac{\varepsilon}{2}$.

Hence, for all $m \geq N$, $d(p, p_m) \leq d(p, p_n) + d(p_n, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $\{p_n\}$ converges to p.

proof of (c))

Any Cauchy sequence is bounded, so a Cauchy sequence in $\mathbb{R}^k$ can be thought of as a Cauchy sequence of some big enough closed ball in $\mathbb{R}^k$. Convergence of such a Cauchy sequence follows from part (b). ∎

Monotonic sequences

Def A sequence $\{s_n\}$ of real numbers is said to be

(a) monotonically increasing if $s_n \le s_{n+1}$ for all $n = 1, 2, \cdots$

(b) monotonically decreasing if $s_n \ge s_{n+1}$ for all $n = 1, 2, \cdots$

Thm Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

proof) Let's suppose $\{s_n\}$ is monotonically increasing (the proof is analogous in the other case)

Let E be the range of $\{s_n\}$.

($\Rightarrow$) Clear, as any convergent sequence is bounded.

($\Leftarrow$) If $\{s_n\}$ is bounded, let $s := \sup E$.

Then, for every $\varepsilon > 0$, there is an integer N such that

$$s - \varepsilon < s_N \le s.$$

Since $\{s_n\}$ is increasing, $s - \varepsilon < s_n \le s$ for all $n \ge N$,

which shows that $\{s_n\}$ converges to s. ∎